

Mathematical Model on Diabetes Millitus Using Fractional Approach; Fractional Homotopy Perturbation Method and Fractional Variational Iteration Method: A Comparison

*Parveen Rana

Associate Professor Department of Mathematics, D.S. College, Aligarh

Abstract

This research considers a mathematical study of the population size of the patient of the diabetes. This study also considers the number of patients with some complications. Through well defining parameter, mathematical model has been categorized into linear or nonlinear. The nonlinear model is presented and threshold value of the population is analyzed for their stability. Modeling equations are simplified using Fractional Variational Iteration Method and Fractional Homotopy Perturbation Method. Figures for validation are obtained using Maple. Considered parameters regarding the population size of diabetes and the number of diabetes patients with complication have largely affect at a time t . Numerical schemes are developed to solve the modeling equations and the numerical simulation is conducted.

Keywords: Fractional Model; Fractional Variational Iteration Method; Diabetes Model; Fractional Homotopy Perturbation Method; Fractional Derivative; Caputo

Article Publication

 Published Online: 20-Feb-2022

*Author's Correspondence

 Parveen Rana

 Associate Professor Department of Mathematics, D.S. College, Aligarh

 parveenrana021076@gmail.com

 [10.53573/rhimrj.2022.v09i02.001](https://doi.org/10.53573/rhimrj.2022.v09i02.001)

© 2022 The Authors. Published by RESEARCH HUB International Multidisciplinary Research Journal. This is an open access article under the CC BY-

NC-ND license 

(<https://creativecommons.org/licenses/by-nc-nd/4.0/>)

Introduction

In several decades, emergence of fraction calculus i.e., order of derivative is fractional works as an important tool in several disciplines of scientific engineering like electromagnetic theory, electric technology, bioengineering, viscoelasticity, control theory, dynamic traffic fluid, neurophysiology, industrial robotics, mathematical economy, potential theory etc. [1-11]. In nature almost all real-world processes are of fractional order. Diffusion of heat into semi-infinite solid in which the half derivative of temperature is considered is good example of fractional order.

The systems with memory are best explained by fractional differential equations. The order in fractional differential equations includes the more degree of freedom, randomness and arbitrariness to result accurate models which ultimately results in better robustness and flexibility in the modeling of signal systems [12]. Moreover, diffusion process in double layer charge is better portrayed in the analysis of fractional order systems. Therefore, the modeling of lithium battery and super capacitors are performed with fractional differential equations. Also, the ceramic bodies characterization and the decay rate of fruits and meats are of the prime importance topics of applications of fractional order differential equations.

There is one more popular area of research to study of earthquake, volcano, and thermo kinetics design and human lungs modeling. Further, economic fluctuations of the market also characterize by fractional differential equation modeling. Based on these several applications of fractional order differential equations, it is observed by several researchers the diabetes modeling has tremendous results using fractional differential equations.

Diabetes is the medication condition of the body failure producing the significant amount of insulin in the body to maintain the sugar level [13]. In most of the cases patients suffer this body failure and recommended taking insulin injection in the body. This is the case of type I diabetes.

However, the type II disease is the health illness which is caused due to unbalanced meal. Losing weight and balance diet are the two major ways of treatment in addition to oral medication. In these patients of type II diabetes heart diseases may strike in the long time [14].

There are two types of diabetes are endemic worldwide: type I and type II. Both type I and type II are insulin dependent at some stage; type I is insulin dependent from initial to final but type II is insulin dependent only when the patient is in need. Type I attack the people below 40 and about 10 to 15 % of the world's population have diabetes. In the initial stage type II identified as non-insulin dependent include the majority of diabetes population about 80-90%. The major cause for diabetes is the spread of obesity. For obesity, unsymmetrical food patterns among the humans are responsible. Several researchers reveals that in the time of approximately ten years, most obese children will have type II diabetes rather type I [15–22].

However, this medical condition is not notifiable to UK health care agencies. The health survey conducted in England reveals that diabetes is prevalent in 3% of the men and 2% in women [23]. In the recent reports of WHO and International Diabetes Federation raise a concern in increasing number of case of diabetes around the world [24,25]. Year 2003 estimates the diabetes prevalence worldwide was still 3%, shows that 190-195 million people are diabetic, living in developing countries [25].

Gestational Diabetes also spread during pregnancy due to the hormonal imbalances and starts from fifth month in pregnancy. However, 45-50 % of women with gestational diabetes get diabetes in their later life [26].

These rates of change are formalized by the ordinary differential equations (ODEs)

$$\begin{aligned}\frac{dD(t)}{dt} &= I - (\lambda + \mu)D + \gamma C, \\ \frac{dC(t)}{dt} &= I + \lambda D - (\gamma + \mu + \nu + \delta)C,\end{aligned}\tag{1}$$

where D indicates the no. of diabetics. C denotes the no. of diabetics having complications. I represent the occurrence of diabetes mellitus.

The fractional diabetes model and it indicates the size of diabetes at the time t , then we get,

$$\begin{aligned}\frac{d^\alpha C(t)}{dt^\alpha} &= -(\lambda + \theta)C + \lambda N, \quad t > 0 \\ \frac{d^\alpha N(t)}{dt^\alpha} &= I - (\nu + \delta)C - \mu N,\end{aligned}\tag{2}$$

where μ denotes the rate of natural mortality, λ indicates the probability of a diabetic person, γ shows the rate of healing complications, ν stands for the rate at which diabetic patients having complication converts critically disabled and δ denotes the mortality rate because of complications.

Fractional variational iteration method (FVIM) [27–30] directly attacks the nonlinear terms and gives result in an infinite series. This method does not require linearization, discretization, perturbations or any restrictive assumptions.

Preliminaries

Definition 2.1. Consider a real function $h(\chi)$, $\chi > 0$. It is called in space C_ζ , $\zeta \in \mathbb{R}$ if $\exists$ a real no. $b (> \zeta)$, s.t. $h(\chi) = \chi^b h_1(\chi)$, $h_1 \in C[0, \infty]$. It is clear that $C_\zeta \subset C_\gamma$ if $\gamma \leq \zeta$.

Definition 2.2. Consider a function $h(\chi)$, $\chi > 0$. It is called in space C_{χ}^m , $m \in \mathbb{N} \cup \{0\}$ if $h^{(m)} \in C_{\chi}$.

Definition 2.3. Left sided Caputo fractional derivative of h , $h \in C_{-1}^m$, $m \in \mathbb{N} \cup \{0\}$,

$$D_t^{\beta} h(t) = \begin{cases} I^{m-\beta} h^{(m)}(t), & m-1 < \beta < m, m \in \mathbb{N}, \\ \frac{d^m}{dt^m} h(t), & \beta = m, \end{cases}$$

a. $I_t^{\zeta} h(x, t) = \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} h(x, s) ds; \zeta, t > 0.$

b. $D_{\tau}^{\nu} V(x, \tau) = I_{\tau}^{m-\nu} \frac{\partial^m V(x, \tau)}{\partial t^m}, m-1 < \nu \leq m.$

c. $D_t^{\zeta} I_t^{\zeta} h(t) = h(t), m-1 < \zeta \leq m, m \in \mathbb{N}.$

d. $I_t^{\zeta} D_t^{\zeta} h(t) = h(t) - \sum_{k=1}^{m-1} h^{(k)}(0^+) \frac{t^k}{k!}, m-1 < \zeta \leq m, m \in \mathbb{N}.$

e. $I^{\nu} t^{\zeta} = \frac{\Gamma(\zeta+1)}{\Gamma(\nu+\zeta+1)} t^{\nu+\zeta}.$

Definition 2.4. Laplace transform of Caputo fractional derivative is

$$L[D^{\alpha} g(t)] = p^{\alpha} F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} g^{(k)}(0), n-1 < \alpha \leq n.$$

Lemma[37]. If u and its partial derivatives are continuous, the fractional derivative $D_t^{\alpha} u(x, y, z, t)$ is bounded.

Basic plan of FVIM for fractional diabetes model

Consider the mathematical model described by Eq. (2) as

$$\frac{d^{\alpha} C(t)}{dt^{\alpha}} = -(\lambda + \theta)C + \lambda N, \quad t > 0$$

$$\frac{d^{\alpha} N(t)}{dt^{\alpha}} = I - (v + \delta)C - \mu N, \quad (3)$$

with initial condition $C(0) = C_0, N(0) = N_0$

A correction functional is built for Eq. (3) as,

$$C_{n+1}(t) = C_n(t) + \int_0^t \lambda \left(\frac{d^{\alpha} C(t)}{d\xi^{\alpha}} + (\lambda + \theta)C - \lambda N \right) (d\xi)^{\alpha},$$

$$N_{n+1}(t) = N_n(t) + \int_0^t \lambda \left(\frac{d^{\alpha} N(t)}{d\xi^{\alpha}} - I + (v + \delta)C + \mu N \right) (d\xi)^{\alpha} \quad (4)$$

where λ is Lagrange's multiplier. By variational theory, λ must satisfy

$$\frac{d^{\alpha} \lambda}{d\xi^{\alpha}} \Big|_{\xi=x} = 0 \quad \text{and} \quad 1 + \lambda \Big|_{\xi=x} = 0.$$

We quickly get, $\lambda = -1$. Then, using it in Eq. (4), we get

$$C_{n+1}(t) = C_n(t) - \int_0^t \left(\frac{d^{\alpha} C(t)}{d\xi^{\alpha}} + (\lambda + \theta)C - \lambda N \right) (d\xi)^{\alpha},$$

$$N_{n+1}(t) = N_n(t) - \int_0^t \left(\frac{d^{\alpha} N(t)}{d\xi^{\alpha}} - I + (v + \delta)C + \mu N \right) (d\xi)^{\alpha}, \quad (5)$$

Finally, we obtain sequences $C_{n+1}(t), N_{n+1}(t), n \geq 0$ of solution. Exact solution is gained as $C(t) = \lim_{n \rightarrow \infty} C_n(t)$ and $N(t) = \lim_{n \rightarrow \infty} N_n(t)$.

Basic plan of FHPTM for fractional diabetes model

To illustrate the process of solution of the FHPTM, we ponder over the system of nonlinear time-fractional PDEs :

$$\frac{d^{\alpha} C(t)}{dt^{\alpha}} = -(\lambda + \theta)C + \lambda N, \quad t > 0 \quad (6)$$

$$\frac{d^{\alpha} N(t)}{dt^{\alpha}} = I - (v + \delta)C - \mu N, \quad (7)$$

$$\text{with initial values} \quad u(x, 0) = h_1(x), v(x, 0) = h_2(x) \quad (8)$$

Here, $\frac{\partial^\alpha}{\partial t^\alpha}$ is (Caputo) fractional derivative of order α , S_1 , S_2 and Q_1 , Q_2 are operators, linear & nonlinear respectively; g_1 and g_2 are the source terms. Also, $0 < \alpha \leq 1$.

Taking Laplace transform on both sides

$$L[D_t^\alpha u(x, t)] + L[S_1(u, v)] + L[Q_1(u, v)] = L[g_1(x, t)] \quad (9)$$

$$L[D_t^\alpha v(x, t)] + L[S_2(u, v)] + L[Q_2(u, v)] = L[g_2(x, t)] \quad (10)$$

By differentiation property of Laplace transform,

$$L[u(x, t)] = p^{-1}h_1(x) - p^{-\alpha}L[g_1(x, t)] + p^{-\alpha}L[S_1(u, v) + Q_1(u, v)] \quad (11)$$

$$L[v(x, t)] = p^{-1}h_2(x) - p^{-\alpha}L[g_2(x, t)] + p^{-\alpha}L[S_2(u, v) + Q_2(u, v)] \quad (12)$$

Taking inverse transform in Eqs. (11) and (12), we get

$$u(x, t) = G_1(x, t) + L^{-1}[p^{-\alpha}L\{S_1(u, v) + Q_1(u, v)\}] \quad (13)$$

$$v(x, t) = G_2(x, t) + L^{-1}[p^{-\alpha}L\{S_2(u, v) + Q_2(u, v)\}] \quad (14)$$

Here $G_1(x, t)$ and $G_2(x, t)$ are the terms coming from the source term and initial values.

Applying HPM, it is assumed that the result may be articulated as a power series,

$$u_n(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) \quad (15)$$

$$v_n(x, t) = \sum_{n=0}^{\infty} p^n v_n(x, t) \quad (16)$$

Here, p is reflected as a small parameter ($p \in [0, 1]$) called homotopy parameter.

The non-linear term is decomposed as

$$Nu(x, t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (17)$$

and,
$$Nv(x, t) = \sum_{n=0}^{\infty} p^n H'_n(v) \quad (18)$$

where H_n and H'_n are He's polynomials of $u_0, u_1, u_2, u_3, \dots, u_n$ and $v_0, v_1, v_2, v_3, \dots, v_n$ respectively. They are calculated by the given formulae:

$$H_n(u_0, u_1, u_2, \dots) = \frac{1}{[n]} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, n = 0, 1, 2, 3, \dots \quad (19)$$

and,
$$H'_n(v_0, v_1, v_2, \dots) = \frac{1}{[n]} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i v_i \right) \right]_{p=0}, n = 0, 1, 2, 3, \dots \quad (20)$$

use Eqs. (10) & (12) in Eq. (8) and Eqs. (11) & (13) in Eq. (9) and applying HPM, we obtain,

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G_1(x, t) + pL^{-1}[p^{-\alpha}L\{S_1(u, v) + Q_1(u, v)\}] \quad (21)$$

$$\text{and,} \quad \sum_{n=0}^{\infty} p^n v_n(x, t) = G_2(x, t) + pL^{-1}[p^{-\alpha}L\{S_2(u, v) + Q_2(u, v)\}] \quad (22)$$

In general, Cherruault and Abbaoui proved the convergence of this above kind of series.

Numerical Implementation of FVIM.

By using conditions, we may initialize with $C(0) = C_0$, $N(0) = N_0$ and using the application of FVIM to Eqs. (2), we get

$$\begin{aligned} C_1(t) &= C_0 - \int_0^t \left(\frac{d^\alpha C_0}{d\xi^\alpha} + (\lambda + \theta)C_0 - \lambda N_0 \right) (d\xi)^\alpha, \\ &= C_0 - \frac{(-(\lambda + \theta)C_0 + \lambda N_0) t^\alpha}{\Gamma(1 + \alpha)}, \end{aligned}$$

(6)

$$\begin{aligned} N_1(t) &= N_0 - \int_0^t \left(\frac{d^\alpha N_0}{d\xi^\alpha} - I + (v + \delta)C_0 + \mu N_0 \right) (d\xi)^\alpha, \\ &= N_0 - \frac{(I - (v + \delta)C_0 - \mu N_0) t^\alpha}{\Gamma(1 + \alpha)}, \end{aligned}$$

(7)

$$\begin{aligned} C_2(t) &= C_1 - \int_0^t \left(\frac{d^\alpha C_1}{d\xi^\alpha} + (\lambda + \theta)C_1 - \lambda N_1 \right) (d\xi)^\alpha, \\ &= C_0 + \frac{((\lambda + \theta)C_0 - \lambda N_0) t^\alpha}{\Gamma(1 + \alpha)} + \frac{((- \delta \lambda + (\theta + \lambda)^2 - \lambda v)C_0 + \lambda (I - (\theta + \lambda + \mu)N_0) t^{2\alpha}}{\Gamma(1 + 2\alpha)}, \end{aligned}$$

(8)

$$\begin{aligned} N_2(t) &= N_1 - \int_0^t \left(\frac{d^\alpha N_1}{d\xi^\alpha} - I + (v + \delta)C_1 + \mu N_1 \right) (d\xi)^\alpha, \\ &= N_0 + \frac{(-I + (v + \delta)C_0 + \mu N_0) t^\alpha}{\Gamma(1 + \alpha)} + \frac{(-I\mu + (\theta + \lambda + \mu)(v + \delta)C_0 + (\mu^2 - \lambda(v + \delta))N_0) t^{2\alpha}}{\Gamma(1 + 2\alpha)}. \end{aligned} \quad (9)$$

Proceeding in this way, the next iteration components can be found with the help of Maple package. At last, we can get solution as

$$\begin{aligned} C(t) &= \lim_{n \rightarrow \infty} C_n(t), \\ N(t) &= \lim_{n \rightarrow \infty} N_n(t). \end{aligned} \quad (10)$$

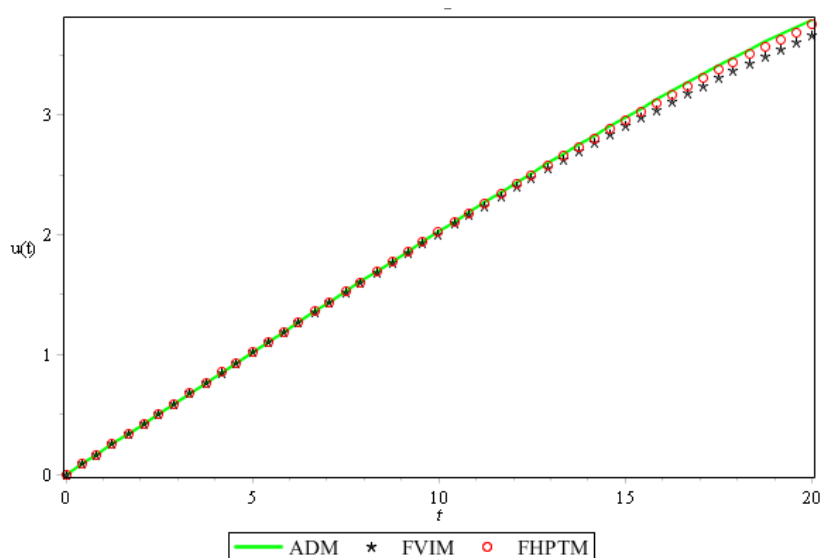


Fig. 1. Behavior of $C(t)$ different value of α

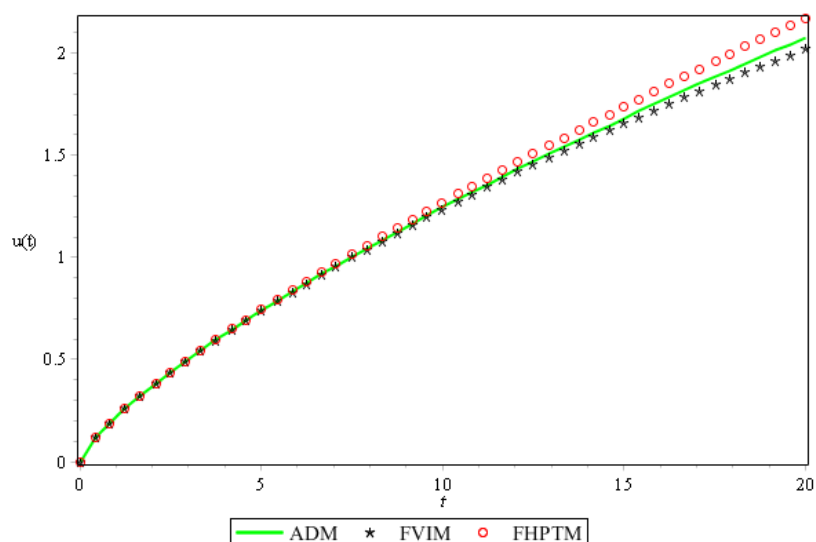


Fig. 2. Behavior of $N(t)$ different value of α

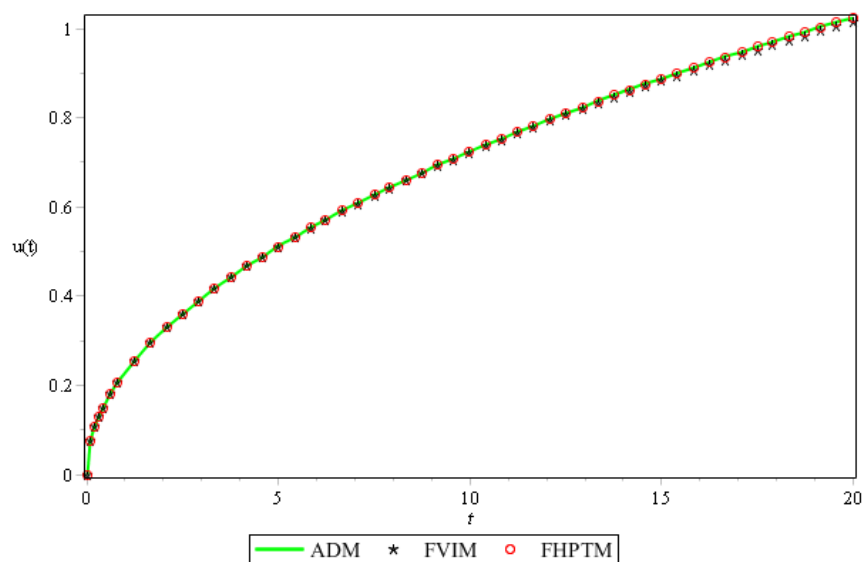


Fig. 3. Behavior of $C(t)$ different value of ν

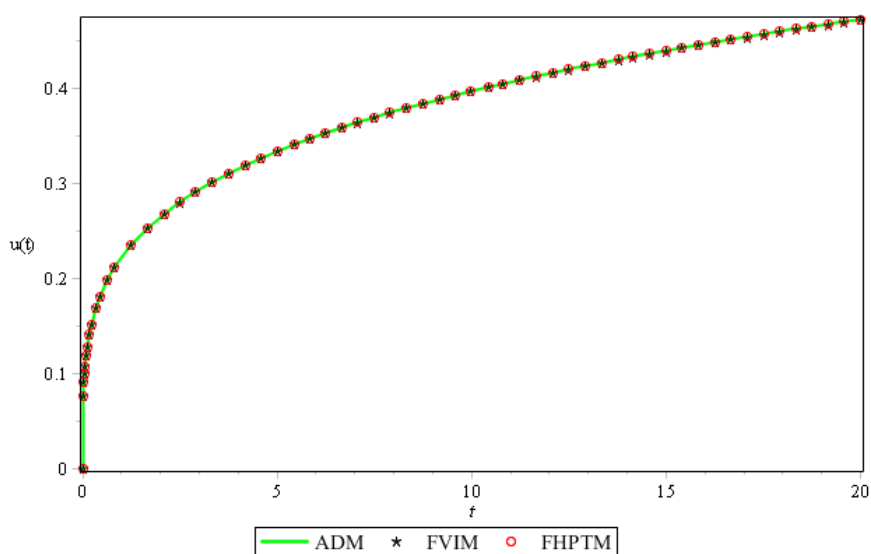


Fig. 4. Behavior of $N(t)$ for ν

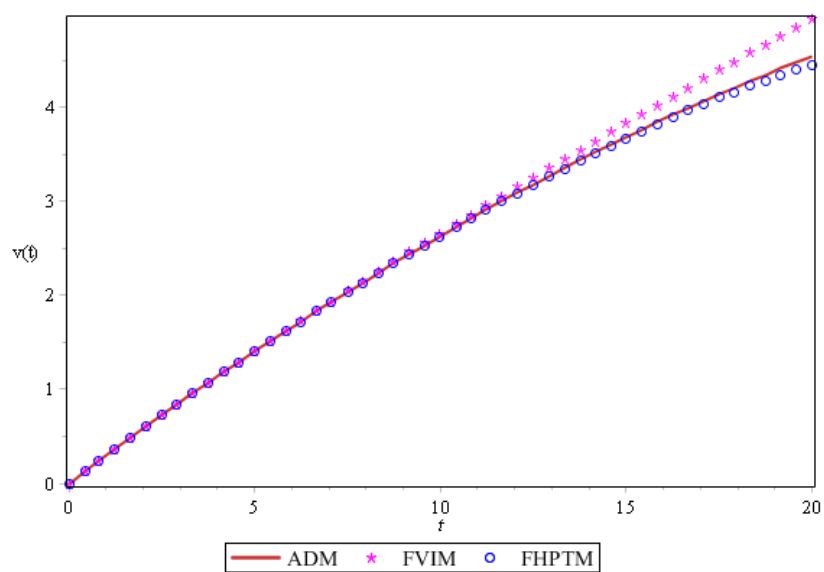


Fig. 5. Behavior of $C(t)$ for δ

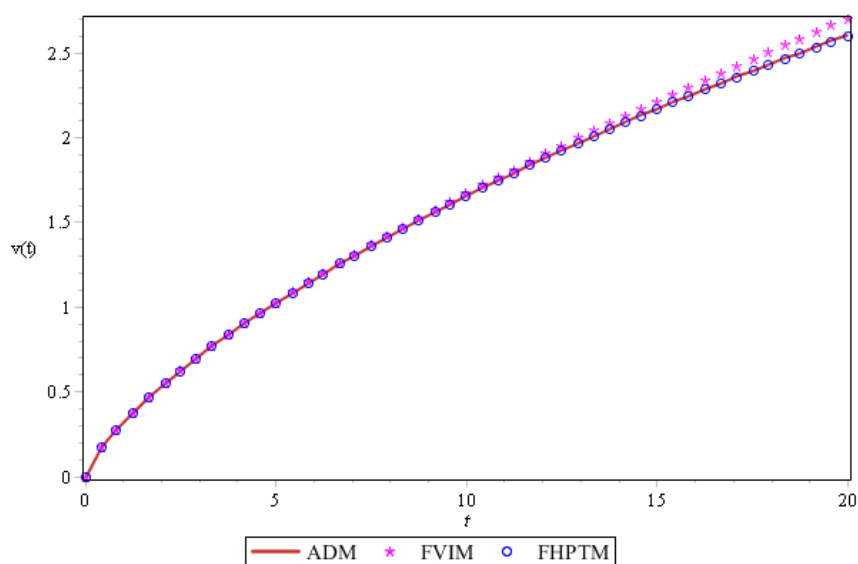


Fig. 6. Behavior of $N(t)$ for δ

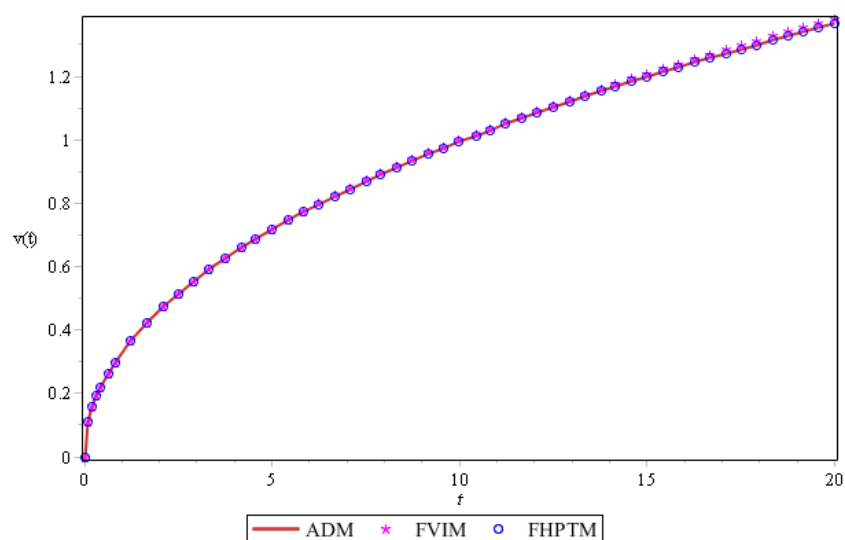


Fig. 7. Behavior of $C(t)$ for μ

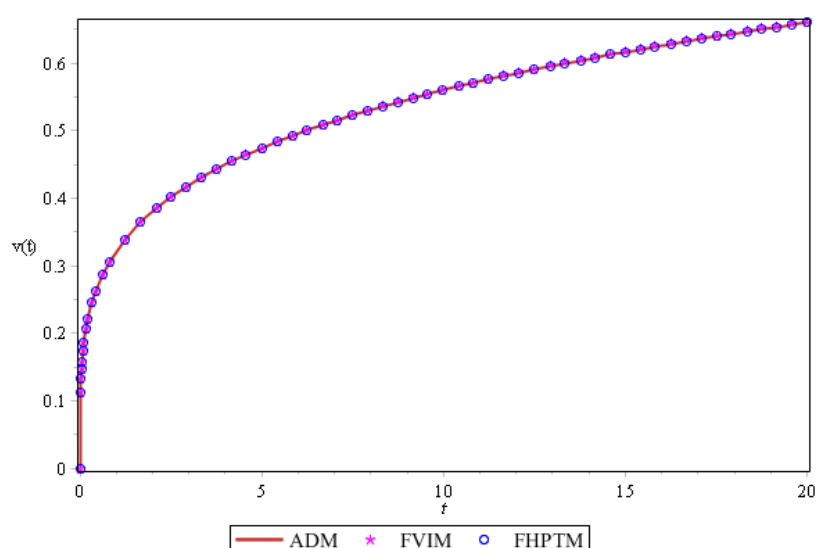


Fig. 8. Behavior of $C(t)$ for δ

Conclusion

Application of fractional Variational Iteration Method; FVIM and Fractional Homotopy Perturbation Method; FHPTM for diabetes model is presented. Both methods are strong numerical computational method for solving nonintegro differential equations. The effect of considering arbitrary order has been observed through numerical method. Various parameters have various complications on the number of diabetes patients with respect to time are portrayed through graphs. The results obtained in this study are very important for diabetes practitioners.

References

- J. H. He, Some applications of nonlinear fractional differential equations and their approximations, *Bull. Sci. Technol. Soc.* 15 (2) (1999) 86–90.
- A. D. Robinson, The use of control systems analysis in neurophysiology of eye movements, *Annu. Rev. Neurosci.* 4 (1981) 463–503.
- R. L. Bagley, P. J. Torvik, Fractional calculus in the transient analysis of viscoelasticity damped structures, *AIAA J.* 23 (1985) 918–925.
- R. L. Magin, Fractional calculus in bioengineering, *Crit. Rev. Biomed. Eng.* 32 (2004) 1–104.
- G. W. Bohannon, Analog fractional order controller in temperature and motor control applications, *J. Vib. Control* 14 (2008) 1487–1498.
- N. Engheta, On the role of fractional calculus in electromagnetic theory, *IEEE Antennas Propag. Mag.* 39 (4) (1997) 35–46.
- G. Cooper, D. Cowan, The application of fractional calculus to potential field data, *Explor. Geophys.* 34 (2003) 51–56.
- M. F. Ali, M. Sharma, R. Jain, An application of fractional calculus in Electrical Engineering, *Adv. Eng. Tec. Appl.* 5 (4) (2016) 41–45.
- R. L. Magin, Fractional calculus models of complex dynamics in biological tissues, *Comput. Math. Appl.* 59 (5) (2010) 1586–1593.
- Q. Xu, J. Huang, L. Zhou, ANN-inversion based fractional-order sliding control for the industrial robot, *Advancing Technology for Humanity* (2015) 4501–4505.
- H. Fallahgoul, S. Focardi, F. Fabozzi, *Fractional Calculus and Fractional Processes with Applications to Financial Economics*, San Diego, CA Elsevier Science 2016.
- R. Panda, M. Dash, Fractional generalized splines and signal processing, *Signal Process.* 86 (2006) 2340–2350.
- B. M. Patil, R. C. Joshi, D. Toshniwal, Hybrid prediction model for type-2 diabetic patients, *Expert Syst. Appl.*, 37, (2010), no. 12, 8102 – 8108.
- C. Zecchin, A. Facchinetti, G. Sparacino, G. De Nicolao, C. Cobelli, A new neural network approach for short-term glucose prediction using continuous glucose monitoring time-series and meal information, *Proceedings of the IEEE International Conference on Engineering Med. Biol. Soc.*, 2011, 5653–5656.

- Boutayeb, A., Twizell, E.H., Achouayb, K., Chetouani, A.: A mathematical model for the burden of diabetes and its complications. *Biomed. Eng. Online* 3(1), 20 (2004). <https://doi.org/10.1186/1475-925X-3-20>
- Zecchin, C., Facchinetti, A., Sparacino, G., De Nicolao, G., Cobelli, C.: A new neural network approach for short-term glucose prediction using continuous glucose monitoring time-series and meal information. *Conf. Proc. IEEE Eng. Med. Biol. Soc.* 2011, 5653–5656 (2011).
- Sharief, A.A., Sheta, A.: Developing a mathematical model to detect diabetes using multigene genetic programming. *Int. J. Adv. Res. in Artif. Intell. (IJARAI)* 3(10), 54 (2014).
- Rosado, Y.C.: Mathematical model for detecting diabetes. In: *Proceedings of the National Conference on Undergraduate Research (NCUR)*, University of Wisconsin La-Crosse, La-Crosse (2009).
- Ackerman, E., Gatewood, I., Rosevear, J., Molnar, G.: Blood glucose regulation and diabetes. In: Heinmets, F. (ed.) *Concepts and Models of Biomathematics*, pp. 131–156. Decker, New York (1969).
- De Gaetano, A., Hardy, T., Beck, B., Abu-Raddad, E., Palumbo, P., Bue-Valleskey, J., Pøksen, N.: Mathematical models of diabetes progression. *Am. J. Physiol. Endocrinol. Metab.* 295, E1462–E1479 (2008) First published September 9, 2008. <https://doi.org/10.1152/ajpendo.90444.2008>
- Patil, B.M., Joshi, R.C., Toshniwal, D.: Hybrid prediction model for type-2 diabetic patients. *Expert Syst. Appl.* 379(12), 8102–8108 (2010)
- Enagi, A.I., Bawa, M., Sani, A.M.: Mathematical study of diabetes and its complication using the homotopy perturbation method. *Int. J. Math. Comput. Sci.* 12(1), 43–63 (2017)
- British Diabetic Association [BDA], *Diabetes in the United Kingdom* (1996).
- International Diabetes Federation [IDF], *IDF Report 2003*, www.idf.org/home/index.cfm
- World Health Organisation [WHO], *The World Health Report 2003: Today's Challenges*, Geneva (2003), <http://www.who.int/whr/2003/en>
- Ahlam A. Sharief, Alaa Sheta, Developing a Mathematical Model to Detect Diabetes Using Multigene Genetic Programming, *International Journal of Advanced Research in Artificial Intelligence*, 3, (2014), no. 10, 54.
- J. H. He, Variational iteration method—a kind of nonlinear analytical technique: Some examples, *Int. J. Nonlinear Mech.* 34 (1999) 699–708.
- A. Prakash and M. Kumar, He's Variational Iteration Method for solution of nonlinear Newell–Whitehead–Segel Equation, *J. Appl. Anal. Comput.* 6 (2016) 738–748.
- J. H. He and X. H. Wu, Variational iteration method: new development and applications, *Comput. Math. Appl.* 54 (2007) 881–894.
- A. Prakash, M. Kumar, K. K. Sharma, Numerical method for solving fractional coupled Burgers equations, *Appl. Math. Comput.* 260 (2015) 314–320.